Comparative study of self-avoiding trails and self-avoiding walks on a family of compact fractals

Ivan Živić,[1] Sava Milošević,[2] and H. Eugene Stanley[3]
[1]Faculty of Natural Sciences and Mathematics, University of Kragujevac, 34000 Kragujevac, Serbia, Yugoslavia
[2]Faculty of Physics, University of Belgrade, P.O. Box 368, 11001 Belgrade, Serbia, Yugoslavia
[3]Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215
(Received 21 May 1998)

We present an exact and Monte Carlo renormalization group (MCRG) study of trails on an infinite family of the plane-filling (PF) fractals, which appear to be compact, that is, their fractal dimension d_f is equal to 2 for all members of the fractal family enumerated by the odd integer b $(3\leqslant b<\infty)$. For the PF fractals, we calculate exactly (for $3\leqslant b\leqslant 7$) the critical exponents ν (associated with the mean squared end-to-end distances of trails) and γ (associated with the total number of different trails). In addition, we calculate ν and γ through the MCRG approach for $b\leqslant 201$ and $b\leqslant 151$, respectively. The MCRG results for $3\leqslant b\leqslant 7$ deviate from the exact results at most 0.04% in the case of ν and 0.14% in the case of γ. Our results show clearly that ν first increases for small values of b (up to $b=9$) and then starts to decrease, resembling the large b behavior of ν for self-avoiding walks (SAWs) on the PF fractals. Similarly, our results show that the trail critical exponent γ, being always larger than the SAW Euclidean value 43/32, monotonically increases with b and for large b displays virtually the same behavior as the corresponding critical exponent γ for SAWs on the PF fractals. We comment on a possible relevance of the comparative study of the criticality of trails and SAWs on the PF family of fractals to the problem of the uniqueness of the universality class for trails and SAWs on the two-dimensional Euclidean lattices, by discussing the fractal-to-Euclidean crossover behavior of ν and γ. [S1063-651X(98)07910-0]

PACS number(s): 05.40.+j, 05.50.+q, 64.60.Ak, 61.41.+e

I. INTRODUCTION

The self-avoiding walk (SAW) on a lattice is a random walk that must not contain self-intersections, which implies that the walker must not cross a site more than once. It has been extensively studied as a challenging problem in statistical physics and, in particular, as a satisfactory model of a linear polymer chain in a good solvent [1]. In the latter case, the forbidden self-intersections of the SAW path correspond to the excluded-volume interactions of monomers that comprise the polymer chain. A random walk model, with a less restrictive excluded-volume interaction, has been introduced [2] under the name of self-avoiding trail, or simply *trail*, for which no lattice bond is allowed to be visited more that once while lattice sites may be revisited. From the geometrical point of view, the lattice trail model has the same relationship to the SAW model as does the bond percolation model to the lattice site percolation model. The criticality of both the SAW and the trail model, that is, their asymptotic properties for a large number N of steps, appears to belong to the category of difficult problems in the critical phenomena studies. In that context, one of the main issues has been whether SAWs and trails belong to the same universality class. However, to answer this question properly it is necessary to obtain reliable results for the critical exponents of trails since the critical exponents for SAWs have been rather firmly established, at least for the two-dimensional Euclidean lattices [3].

To learn critical exponents of trails on the Euclidean lattices, various approaches have been applied, including rigorous analysis [4], exact enumeration techniques [5,6], momentum space renormalization group (RG) and the ϵ expansion [7], Monte Carlo (MC) studies, and scanning simulation methods [8–10], and transfer-matrix studies [11,12]. In spite of the numerous studies, few exact results for trails have been obtained. For this reason, it is desirable to study a family of fractal lattices whose members allow, in principle, an exact treatment of the problem and whose characteristics approach (via the fractal-to-Euclidean crossover) properties of a Euclidean lattice. In addition, it is desirable to accomplish the latter task on a family of fractals for which the SAW problem can be well analyzed.

In this paper we report an exact RG study and the Monte Carlo renormalization group (MCRG) study of trails on an infinite family of plane-filling (PF) fractals that appear to be *compact*, that is, their fractal dimension d_f is equal to 2 for all members of the fractal family enumerated by the odd integer b $(3\leqslant b<\infty)$. For the PF fractals, we calculate exactly and through the MCRG approach the critical exponents ν (associated with the mean squared end-to-end distances of trails) and γ (associated with the total number of different trails). We perform our calculations for as many members of the fractal family as possible in order to study the behavior of the critical exponents in the fractal-to-Euclidean crossover region, which asymptotically appears when $b\to\infty$. For the sake of comparison of the obtained results for trails with those of SAWs, we extend here the set of data that has been previously found in a study of SAWs on the same family of fractals [13].

The present paper is organized as follows. We define the PF family of fractals in Sec. II, where we also present the framework of our exact and MCRG approach to the evaluation of the critical exponents ν and γ of trails on the PF fractals, together with some specific results. In Sec. III we compare the critical exponents ν and γ for the trails and SAWs and present pertinent conclusions.

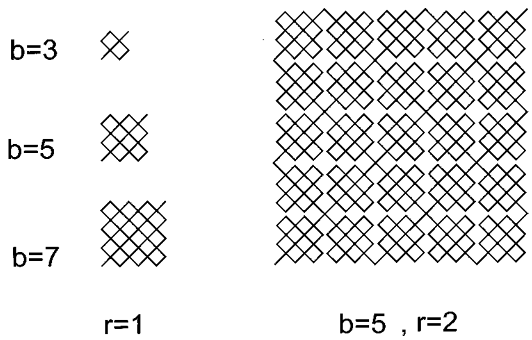

FIG. 1. First three fractal generators ($r=1$) of the plane-filling (PF) family of fractals and the second stage ($r=2$) of the $b=5$ PF fractal.

II. TRAILS ON THE PLANE-FILLING FRACTAL LATTICES

In this section we apply the exact RG and the MCRG method to calculate asymptotic properties of trails on the PF fractal lattices. Each member of the PF fractal family is labeled by an odd integer b ($3\leqslant b<\infty$) and can be constructed in stages. At the initial stage ($r=1$) the lattices are represented by the corresponding generators (see Fig. 1). The rth stage fractal structure can be obtained iteratively in a self-similar way, that is, by enlarging the generator by a factor b^{r-1} and by replacing each of its segments with the $(r-1)$th stage structure (see Fig. 1), so that the complete fractal is obtained in the limit $r\to\infty$. The shape of the fractal generators and the way the fractals are constructed imply that each member of the family has fractal dimension d_f equal to 2. Thus they appear to be compact objects (with no voids) embedded in a two-dimensional Euclidean space, that is, they resemble square lattices with various degrees of inhomogeneity distributed self-similarly.

The basic asymptotic properties of trails, analogously to the case of SAWs, are described by two critical exponents ν and γ. The critical exponent ν is associated with the scaling law $\langle R_N^2\rangle\sim N^{2\nu}$ for the mean squared end-to-end distance for N-step trails, whereas the critical exponent γ is associated with the scaling law $C_N\sim\mu^N N^{\gamma-1}$ for the total number C_N of distinct trails of N steps (averaged over all possible positions of the starting point). Here μ is the trail connectivity constant and it is assumed that N is a very large number. We calculate these critical exponents in the framework of the RG method, in which we study the corresponding generating functions that can be defined by introducing the weight factor x (fugacity) for each step of the trail. The generating functions

$$C(x)\equiv\sum_{N=1}^{\infty} C_N x^N \tag{1}$$

and

$$L(x)\equiv\sum_{N=2}^{\infty}\langle R_N^2\rangle C_N x^N/C(x), \tag{2}$$

whose leading singular terms, when x approaches $1/\mu$ from below, are of the form

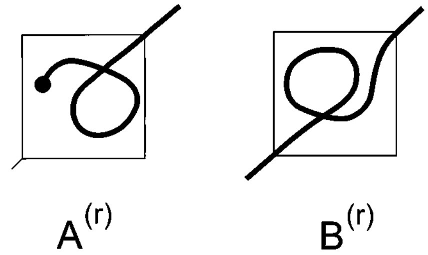

FIG. 2. Diagrammatic representation of the two restricted partition functions for an rth stage of the fractal construction of a member of the PF family. The fractal interior structure is not shown. Thus, for example, $A^{(r)}$ represents the trail that starts somewhere within the rth stage fractal structure and leaves it at its upper right link to rest of fractal.

$$C(x)\sim(1-x\mu)^{-\gamma} \tag{3}$$

and

$$L(x)\sim(1-x\mu)^{-2\nu}. \tag{4}$$

In order to calculate ν and γ we have found that it is useful to introduce two restricted partition functions $A^{(r)}$ and $B^{(r)}$ (see Fig. 2). The two restricted partition functions represent partial sums of statistical weights of all possible trails within the rth stage fractal structure for the two kinds of trails depicted in the Fig. 2. The corresponding initial conditions

$$A^{(0)}=\sqrt{x},\quad B^{(0)}=x \tag{5}$$

are relevant to the fractal unit segment ($r=0$). For arbitrary r, the self-similarity of the fractals under study implies the recursion relations

$$A^{(r)}=f_A(A^{(r-1)},B^{(r-1)}) \tag{6}$$

and

$$B^{(r)}=f_B(B^{(r-1)}), \tag{7}$$

where the explicit forms of the functions f_A and f_B are (due to the underlying self-similarity) independent of the specific value of r. These equations comprise the renormalization group for the trail problem.

We start by applying the above RG framework to find the trail critical exponent ν for the PF fractals. First we shall present the corresponding exact calculation and then we shall present the MCRG approach. To this end, we need to analyze Eq. (7) at the corresponding fixed point. It can be shown that f_B is a simple polynomial, so from Eq. (7)

$$B'=\sum_N a_N B^N, \tag{8}$$

where we have used the prime for rth-order partition function and no indices for the $(r-1)$th-order partition function. Here one should observe that the foregoing RG set of equations has the same general form in the case of SAWs and in the case of trails. The specific differences between the two cases appear in the values of the coefficients a_N, which are the numbers of all possible trails (SAWs) of N steps that traverse the fractal generator.

Knowing the RG equation (8), the critical exponent ν follows from

$$\nu = \frac{\ln b}{\ln \lambda_1}, \tag{9}$$

where λ_1 is the relevant eigenvalue of the RG equation (8) at the nontrivial fixed point $0 < B^* < 1$ [13,14], that is,

$$\lambda_1 = \left. \frac{dB'}{dB} \right|_{B^*}. \tag{10}$$

Consequently, evaluation of ν starts with determining the coefficients a_N of Eq. (8) and finding the pertinent fixed point value B^*, which is, according to the initial conditions (5), equal to the critical fugacity $x^* = 1/\mu$. In the case of trails, we have been able to find exact values of a_N for $3 \leqslant b \leqslant 7$, which are given in the Appendix (whereas in the case of SAWs we reported the corresponding values for $3 \leqslant b \leqslant 9$ in the Appendix of [13]).

Comparing the two cases (trails versus SAWs), one can see that a_N for trails are always bigger than a_N for SAWs, which indicates that the case of exact enumeration of all possible trails is more difficult than the enumeration of SAWs. This difference springs from the definition of the two kinds of walks wherefrom it follows that SAWs, for a given number of steps, comprise a subset of trails. Knowing a_N, for a given b, we apply Eqs. (8)–(10) to learn the critical fugacity x^* (that is, B^*) and the critical exponent ν. Our results for B^* and ν, for $b = 3,5$, and 7, are (0.654 93, 0.686 50), (0.554 15, 0.716 52), and (0.503 04, 0.720 72), respectively.

To overcome the computational problem of learning exact values of a_N, we apply the MCRG method for $b \geqslant 9$. It has been justified in a number of cases [13,15,16] that, due to both the inherent self-similarity and the finite ramification of the underlying lattices, this method should work better in the case of fractals than in the case of regular lattices. The essence of the MCRG method [13,15] consists of treating B', given by Eq. (8), as the grand canonical partition function that comprise all possible trails (SAWs) that traverse the fractal generator at two fixed apices. In this spirit, Eq. (8) allows us to write the relation

$$\frac{dB'}{dB} = \frac{B'}{B} \langle N(B) \rangle, \tag{11}$$

where $\langle N(B) \rangle$ is given by

$$\langle N(B) \rangle = \frac{1}{B'} \sum_N N a_N B^N, \tag{12}$$

which can be considered as the average number of steps, made at fugacity B, by all possible trails (SAWs) that cross the fractal generator. Comparing Eq. (10) with Eq. (11) we obtain the equality $\lambda_1 = \langle N(B^*) \rangle$ and thereby we obtain

$$\nu = \frac{\ln b}{\ln \langle N(B^*) \rangle}. \tag{13}$$

TABLE I. MCRG results for trails on the PF fractals enumerated by the scaling parameter b. The corresponding MCRG fixed point values B^* and the critical exponents ν and γ are given in the second, third, and fourth columns, respectively.

b	B^*	ν	γ
3	0.65485±0.00018	0.68674±0.00024	1.4274±0.0020
5	0.55401±0.00013	0.71654±0.00015	1.5884±0.0023
7	0.50315±0.00009	0.72045±0.00012	1.6468±0.0025
9	0.47320±0.00008	0.72098±0.00010	1.6847±0.0028
11	0.45390±0.00006	0.72054±0.00009	1.7219±0.0030
13	0.44041±0.00006	0.72016±0.00009	1.7554±0.0033
15	0.43035±0.00016	0.71932±0.00026	1.7792±0.0039
17	0.42276±0.00005	0.71879±0.00008	1.8025±0.0037
21	0.41157±0.00012	0.71798±0.00023	1.8365±0.0045
25	0.40413±0.00010	0.71662±0.00021	1.8732±0.0049
31	0.39669±0.00006	0.71530±0.00014	1.9257±0.0053
35	0.39290±0.00008	0.71478±0.00019	1.9388±0.0059
41	0.38888±0.00002	0.71372±0.00004	1.9775±0.0060
51	0.38416±0.00005	0.71278±0.00012	2.0090±0.0070
61	0.38130±0.00006	0.71157±0.00016	2.0499±0.0080
71	0.37903±0.00006	0.71050±0.00015	2.0791±0.0085
81	0.37745±0.00005	0.71021±0.00014	2.1115±0.0092
91	0.37662±0.00005	0.70938±0.00014	2.1251±0.0115
101	0.37517±0.00004	0.70929±0.00014	2.1341±0.0098
121	0.37373±0.00004	0.70817±0.00012	2.1245±0.0092
151	0.37236±0.00002	0.70772±0.00008	2.1475±0.0159
171	0.37170±0.00003	0.70716±0.00011	
201	0.37100±0.00003	0.70595±0.00011	

This is the formula that enables us to calculate ν via the MCRG method, that is, without calculating explicitly the coefficients a_N.

For a given fractal (with scaling factor b), we begin by determining the critical fugacity B^*. To this end, we start the Monte Carlo simulation with an initial guess for the fugacity B_0 in the region $0 < B_0 < 1$. Here B_0 can be interpreted as the probability of making the next step along an available direction from the vertex that the walker has reached. Let us assume that S_0 is the total number of the MC simulations of walks (at the chosen B_0) and let S_0^t of them be those that traverse the fractal generator. Hence the ratio S_0^t/S_0 is the renormalized fugacity B_0' of the coarse-grained fractal structure. In this way we obtain the value of the sum (8) without specifying the set a_N. Then the next values B_n $(n \geqslant 1)$ at which the MC simulation should be performed can be found by using the ''homing'' procedure [17], which can be closed at the stage when the difference $B_n - B_{n-1}$ becomes less than the statistical uncertainty associated with B_{n-1}. Consequently, B^* can be identified with the last value B_n found in this way. Performing the MC simulation at the value B^*, we can record all possible trails (SAWs) that traverse the fractal generator. Then, knowing such a set of walks, we can represent the average value of the length of a walk (that traverses the generator) via the corresponding average number of steps $\langle N(B^*) \rangle$. Accordingly, we can learn the value of the critical exponent ν through the formula (13). In Table I we present our MCRG results for the trail critical exponent ν, together with the related critical fugacity B^*, for the PF fractal lattices with $b \leqslant 201$. Here we note that, comparing the MCRG results for $3 \leqslant b \leqslant 7$ with the exact results reported above, we can see that there is no deviation larger than 0.04%.

We now apply the RG method to find the trail critical exponent γ for the PF fractals, which determines the singular part (3) of the generating function $C(x)$ defined by Eq. (1). To learn the singular behavior of $C(x)$, in the vicinity of B^*, one needs to know the corresponding behavior of the restricted partition functions $A^{(r)}$ and $B^{(r)}$ [13]. Since in the previous paragraphs we have learned the behavior of $B^{(r)}$, it remains to analyze here the recursion relations (6). The configuration of possible trail paths imply the following structure of the recursion relation

$$A^{(r)} = a(B^{(r-1)})A^{(r-1)}, \tag{14}$$

where $a(B^{(r-1)})$ is a polynomial in $B^{(r-1)}$. This formula allows us to find the critical exponent γ. Hence we first note that in the case under study, according to the procedure detailed in previous papers [13,14,18], γ can be expressed in the form

$$\gamma = 2\frac{\ln(\lambda_2/b)}{\ln \lambda_1}, \tag{15}$$

where

$$\lambda_2 = a(B^*) \tag{16}$$

is the RG eigenvalue of the polynomial $a(B^{(r-1)})$ defined by Eq. (14), with B^* being the fixed point value of Eq. (8). Therefore, it remains either to find means to determine exactly an explicit expression for the polynomial $a(B^{(r-1)})$ or to surpass this step and to evaluate only the single needed value $a(B^*)$.

In order to obtain an explicit expression of the polynomial $a(B^{(r-1)})$, we note that its form, due to the underlying self-similarity of the PF fractals, should not depend on r and so in what follows we assume $r=1$. We then can verify the expression

$$a(B) = 1 + \sum_N Q_N B^N, \tag{17}$$

where Q_N is the number of all trails (SAWs) of N steps that start at any bond within the generator ($r=1$) and leaves it at a fixed exit, which implies that the above sum starts with the $N=1$ term. By enumeration of all relevant walks, the coefficients Q_N can be evaluated exactly up to certain finite b. In the Appendix we present specific values of Q_N for $3 \leqslant b \leqslant 7$. Using the data given in the Appendix, together with Eqs. (15)–(17) (and previously found B^* and λ_1), we have obtained the values $\gamma = 1.429\,40$ (for $b=3$), $\gamma = 1.589\,44$ (for $b=5$), and $\gamma = 1.646\,20$ (for $b=7$).

For a sequence of $b \geqslant 9$, the exact determination of the polynomial (17) (that is, knowledge of the coefficients Q_N) can be hardly reached using the present-day computers. However, to calculate λ_2 one does not need a complete knowledge of polynomial $a(B)$. In fact, to obtain λ_2, one needs only values of this polynomial at the fixed point [see Eq. (16)]. On the other hand, the polynomial that appears in Eq. (14) can be considered to be a grand partition function of an appropriate ensemble and, consequently, within the MCRG method [13,19] the requisite value of the polynomial can be determined directly. Details of the way to ascertain values of $a(B^*)$ are quite similar to the way applied previously [13,19], and here we are not going to elaborate on it further. Owing to the fact that we can obtain $a(B^*)$ through the MC simulations and knowing λ_1 from the preceding calculation of ν, we can apply Eqs. (15) and (16) to learn γ. In Table I we present our MCRG results for γ for $3 \leqslant b \leqslant 151$. Hence, comparing the MCRG values for γ from Table I, for $b=3$, 5, and 7, with the exact results found in this work, one can see that the MCRG values deviate at most 0.14% from the available exact values.

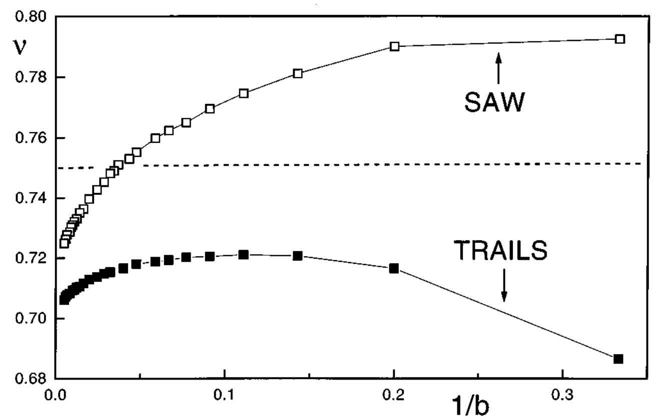

FIG. 3. Results for the critical exponent ν for trails (solid squares), obtained in this work, and results for SAWs (open squares), obtained in Ref. [13] and supplemented in this work for $b=151$, 171, and 201. The horizontal broken line represents the Euclidean value $\nu=3/4$. The solid lines serve as guides to the eye.

III. DISCUSSION AND SUMMARY

We have studied critical properties of trails on the infinite family of the PF fractals whose each member has fractal dimension d_f equal to the Euclidean value 2. In particular, we have calculated the trails' critical exponents ν and γ via an exact RG (for $3 \leqslant b \leqslant 7$) and via the MCRG approach (up to $b \leqslant 201$ for ν and up to $b \leqslant 151$ for γ). Specific results for the trails' critical exponents are presented in Table I. To compare our results with the corresponding results for SAWs on the same family fractals, we have first extended the known [13] sequence of results ($3 \leqslant b \leqslant 121$) for the critical exponent ν and the critical fugacity B^* for SAWs by calculating these two quantities for $b=151$, 171, and 201 via the MCRG method. The corresponding results for (B^*,ν) are (0.384 76, 0.727 75), (0.383 99, 0.726 31), and (0.383 16, 0.724 86). Consequently, in Fig. 3 we show ν to be a function of $1/b$ for both SAWs and trails. Notice that ν for SAWs ($\nu_{\rm SAW}$) in the region of b studied is always larger than ν for trails ($\nu_{\rm trail}$), which implies that the mean end-to-end distance for SAWs is always larger than the mean end-to-end distance for trails

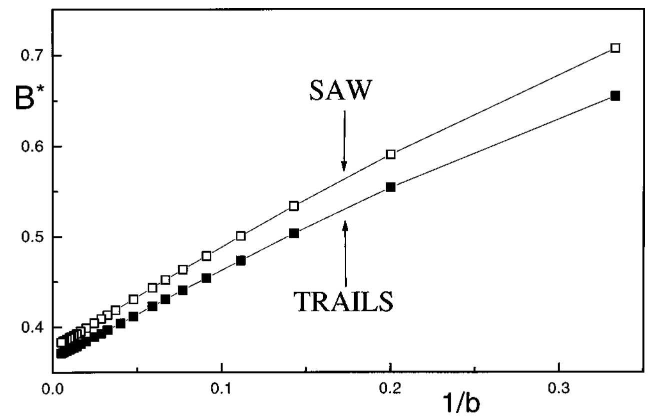

FIG. 4. Results for the fixed point values B^* (the reciprocal connectivity) for trails (solid squares), obtained in this work, and data for SAWs (open squares), obtained in Ref. [13] and supplemented in this work for $b=151$, 171, and 201. The solid lines serve as guides to the eye. As regards the limiting values for $b\to\infty$ and their relation to the Euclidean values, see the text.

$$\langle R_{\rm SAW}\rangle > \langle R_{\rm trail}\rangle. \tag{18}$$

The above relation arises from the fact that in the case of trails the walker can cross twice a large number of the lattice sites, thereby making its path more packed. Next it appears (see Fig. 3) that the trails critical exponent $\nu_{\rm trail}$, being always smaller than the Euclidean value 3/4, is a nonmonotonic function the scaling parameter b. However, one can notice that, for large b, the behavior of $\nu_{\rm trail}$ becomes similar to the behavior of $\nu_{\rm SAW}$, that is, both of them display a monotonic decrease with b. Here we encounter, as it might be expected, the question as to what happens with both critical exponents ($\nu_{\rm SAW}$ and $\nu_{\rm trail}$) in the fractal-to-Euclidean crossover when $b\to\infty$. According to the finite-size scaling arguments [13,14], the critical exponent $\nu_{\rm SAW}$ approaches, from below, the Euclidean value 3/4 when $b\to\infty$. From the above comparison and on the grounds of the established universality of trails and SAWs on the Euclidean lattices [4,5,8,9,11,12], one may suppose that $\nu_{\rm trail}$ will display the same behavior in the limit $b\to\infty$, but this assumption should be a topic of future investigations.

Continuing the comparison of the criticality of SAWs and trails on the PF fractals, we show in Fig. 4 our results for the critical fixed points B^* (which are equal to the reciprocal of the connectivity constant μ) for both types of walks. We observe that $B^*_{\rm SAW}$ are always larger than $B^*_{\rm trail}$, that is, $\mu^*_{\rm SAW}<\mu^*_{\rm trail}$, which is expected because the trail walk has by definition more possibilities to continue walking from a given site. However, we also observe (see Fig. 4) that $B^*_{\rm SAW}$ and $B^*_{\rm trail}$ behave quite similarly as functions of $1/b$ and that these functions become almost linear for large b. This allows us to estimate the limiting values of $B^*_{\rm SAW}$ and $B^*_{\rm trail}$ for $b\to\infty$, which should be compared with the corresponding values for the square lattice. Our detailed numerical analysis reveals that $B^*_{\rm SAW}$ has the asymptotic value $0.379\,15\pm0.000\,40$, which should be compared with the Euclidean value 0.379 052 3(3) for the square lattice, obtained in [20]. Similarly, in the case of trails on the PF fractals, we have found that $B^*_{\rm trail}$ has the limiting value $0.367\,35\pm0.000\,04$,

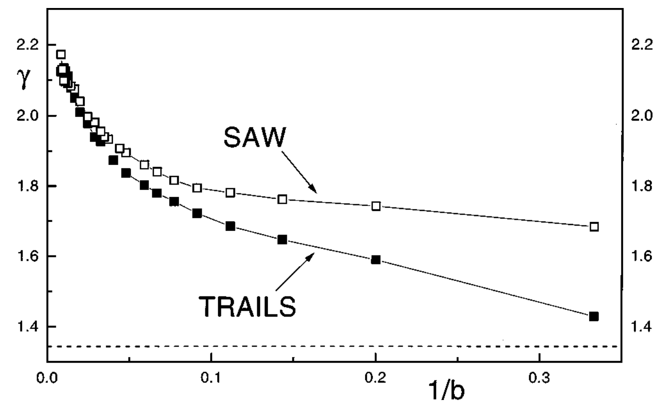

FIG. 5. Results for the critical exponent γ for trails (solid squares), obtained in this work, and data for SAWs (open squares), obtained in Ref. [13]. The horizontal broken line represents the Euclidean value $\gamma=43/32$ for a two-dimensional lattice.

which is in a good agreement with the corresponding result $0.367\,57\pm0.000\,01$ for the square lattice [12].

To complete our comparison of the two types of random walks (trails and SAWs) on the PF fractals, we plot the corresponding values of the critical exponent γ as functions of the fractal enumerator b (see Fig. 5). One can see that in both cases γ, being always larger than the Euclidean value 43/32 [3], monotonically increases with b. In the case of SAWs we demonstrated [13], through the finite-size scaling argument, that γ will continue to increase with b, approaching the non-Euclidean value 103/32 in the limit $b\to\infty$. From Fig. 5 we see that the difference between the two sets (for trails and SAWs) becomes smaller with increasing b and so we expect similar asymptotic behavior in the region of very large b. However, to test such an expectation (which is similar to the case of the critical exponent ν) would require much new work, including the invention of the pertinent finite-size scaling arguments. Finally, as regards Fig. 5, we note that the inequality $\gamma_{\rm SAW}>\gamma_{\rm trail}$ does not imply that the number of SAWs, for a given large number of steps N, can be larger than the number of trails. This observation arises from the previously established inequality $\mu_{\rm SAW}<\mu_{\rm trail}$ (see Fig. 4) and from the power law behavior $\mu^N N^{\gamma-1}$ (for the number of walks) expected to be valid in both cases.

In conclusion, our comparative study of trails and SAWs on the PF family of fractals shows that the two types of random walks display similar critical properties: similar behaviors of the critical exponents (ν and γ) and the connectivity constant, as functions of the fractal scaling parameter b. In addition, the observed similarity becomes more pronounced for large b, that is, close to the fractal-to-Euclidean crossover region ($b\to\infty$), which in a way corroborates the current inference [4,5,8,9,11,12] that SAWs and trails (on the two-dimensional Euclidean lattices) belong to the same universality class.

APPENDIX: COEFFICIENTS OF THE RG TRANSFORMATION

We present coefficients of the RG transformations that have been used to calculate the critical exponents ν and γ for the trails on the PF family of fractals. First, we give the coefficients a_N that appear in the RG relation (8):

$$b=3,\quad a_3=1,\quad a_5=2,\quad a_9=6,$$

$$b=5,\quad a_5=1,\quad a_7=12,\quad a_9=20,\quad a_{11}=62,$$

$$a_{13}=138,\quad a_{15}=186,\quad a_{17}=416,\quad a_{19}=198,$$

$$a_{21}=1056,\quad a_{25}=2592,$$

$$b=7,\quad a_7=1,\quad a_9=30,\quad a_{11}=182,\quad a_{13}=598,$$

$$a_{15}=2362,\quad a_{17}=6960,\quad a_{19}=22\,180,\quad a_{21}=59\,396,$$

$$a_{23}=144\,364,\quad a_{25}=323\,354,\quad a_{27}=654\,690,$$

$$a_{29}=1\,273\,764,\quad a_{31}=2\,068\,716,\quad a_{33}=3\,536\,168,$$

$$a_{35}=4\,747\,076,\quad a_{37}=9\,159\,256,\quad a_{39}=8\,367\,376,$$

$$a_{41}=22\,322\,808,\quad a_{43}=10\,525\,376,$$

$$a_{45}=49\,701\,344,\quad a_{49}=83\,090\,912.$$

In what follows we present the coefficients Q_N of the RG relation (17):

$$b=3,\quad Q_1=3,\quad Q_2=5,\quad Q_3=4,\quad Q_4=8,$$

$$Q_5=6,\quad Q_6=6,\quad Q_7=6,\quad Q_8=6,$$

$$b=5,\quad Q_1=3,\quad Q_2=5,\quad Q_3=15,\quad Q_4=33,$$

$$Q_5=52,\quad Q_6=112,\quad Q_7=160,\quad Q_8=300,\quad Q_9=436,$$

$$Q_{10}=736,\quad Q_{11}=894,\quad Q_{12}=1362,\quad Q_{13}=1520,$$

$$Q_{14}=2140,\quad Q_{15}=2250,\quad Q_{16}=2770,$$

$$Q_{17}=2630,\quad Q_{18}=3430,\quad Q_{19}=3296,\quad Q_{20}=3648,$$

$$Q_{21}=2592,\quad Q_{22}=2592,\quad Q_{23}=2592,\quad Q_{24}=2592,$$

$$b=7,\quad Q_1=3,\quad Q_2=5,\quad Q_3=15,\quad Q_4=33,$$

$$Q_5=91,\quad Q_6=209,\quad Q_7=444,\quad Q_8=1020,$$

$$Q_9=1930,\quad Q_{10}=4310,\quad Q_{11}=7764,$$

$$Q_{12}=16\,580,\quad Q_{13}=29\,010,\quad Q_{14}=57\,942,$$

$$Q_{15}=96\,872,\quad Q_{16}=183\,248,\quad Q_{17}=292\,676,$$

$$Q_{18}=527\,616,\quad Q_{19}=798\,676,\quad Q_{20}=1\,362\,976,$$

$$Q_{21}=19\,657\,00,\quad Q_{22}=3\,190\,292,$$

$$Q_{23}=4\,376\,352,\quad Q_{24}=6\,738\,484,$$

$$Q_{25}=8\,827\,122,\quad Q_{26}=12\,960\,066,$$

$$Q_{27}=16\,118\,644,\quad Q_{28}=22\,632\,952,$$

$$Q_{29}=26\,977\,068,\quad Q_{30}=36\,086\,864,$$

$$Q_{31}=41\,361\,132,\quad Q_{32}=53\,569\,348,$$

$$Q_{33}=59\,243\,924,\quad Q_{34}=73\,567\,212,$$

$$Q_{35}=78\,272\,896,\quad Q_{36}=97\,043\,904,$$

$$Q_{37}=98\,256\,232,\quad Q_{38}=114\,855\,912,$$

$$Q_{39}=114\,120\,664,\quad Q_{40}=136\,780\,568,$$

$$Q_{41}=119\,775\,488,\quad Q_{42}=129\,361\,024,$$

$$Q_{43}=120\,561\,856,\quad Q_{44}=132\,792\,256,$$

$$Q_{45}=83\,090\,912,\quad Q_{46}=83\,090\,912,$$

$$Q_{47}=83\,090\,912,\quad Q_{48}=83\,090\,912.$$

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